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Factorial ratios, hypergeometric series, and a family of step functions

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Abstract

We give a complete classification of a certain family of step functions related to the Nyman–Beurling approach to the Riemann hypothesis and previously studied by Vasyunin. Equivalently, we completely describe when certain sequences of ratios of factorial products are always integral. Essentially, once certain observations have been made, this comes down to an application of Beukers and Heckman's classification of the monodromy of the hypergeometric function ${}_nF_{n-1}$. We also note applications to the classification of cyclic quotient singularities.

1. *Introduction*

In [**17**], Vasyunin considered the following problem, originating from the Nyman–Beurling formulation of the Riemann hypothesis.

Problem 1.1. Classify all step functions of the form

$$f(x) = \sum_{i=1} c_i \left\lfloor \frac{x}{m_i} \right\rfloor, \quad m_i \in \mathbb{N}, \;\; c_i \in \mathbb{Z}, \tag{1.1}$$

having the property that $f(x) \in \{0, 1\}$ for all x.

Vasyunin discovered some infinite families of solutions and 52 additional sporadic solutions and, based on the results of extensive computations, formulated conjectures ([**17**, Conjectures 8 and 11]) that these lists were complete.

It follows easily from a theorem of Landau (see Lemma 3.2) that this problem is equivalent to the following problem.

Problem 1.2. Let $\mathbf{a} \in \mathbb{N}^K, \mathbf{b} \in \mathbb{N}^{K+1}$, and set

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_{K+1} n)!}.$$

For what parameters $\mathbf{a}$ and $\mathbf{b}$ with $\sum a_i = \sum b_j$ is $u_n(\mathbf{a}, \mathbf{b})$ an integer for all n?

It is immediately evident that in Problem 1.1 we may make a change of variables $m_i' = C m_i$ without changing the output of the function. Similarly, though not as immediately obvious, $(\mathbf{a}, \mathbf{b})$ is a solution to Problem 1.2 if and only if $((1/d)\mathbf{a}, (1/d)\mathbf{b})$ is a solution, where $d = \gcd(\mathbf{a}, \mathbf{b})$. Thus, in both cases it is enough to classify solutions with a greatest common divisor of 1, as all other solutions are multiples of these.

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In [15], Rodriguez-Villegas observed that the generating function attached to $u_n(\mathbf{a}, \mathbf{b})$, given by

$$u(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n,$$

is in fact a hypergeometric series, and that $u_n(\mathbf{a}, \mathbf{b})$ is always integral if and only if $u(\mathbf{a}, \mathbf{b}; z)$ is an algebraic function. This observation, which is explained in Section 4, allows us to use the work of Beukers and Heckman [6] to give a complete answer to Problem 1.2 and to prove Vasyunin's conjectures about the completeness of his classification of such step functions.

Specifically, we prove the following theorem, which asserts that Conjectures 8 and 11 in [17] are true.

THEOREM 1.3. *Let*

$$f(x) = \sum_{k=1}^{N} \left\lfloor \frac{x}{m_k} \right\rfloor - \sum_{k=N+1}^{2N+1} \left\lfloor \frac{x}{m_k} \right\rfloor$$

and suppose that $m_i \neq m_j$ for all $i \leqslant N$ and $j \leqslant N+1$, and that

$$\gcd(m_1, m_2, \ldots, m_{2N+1}) = 1.$$

Then $f(x)$ takes only the values 0 and 1 if and only if either of the following statements hold:

(i) *$f(x)$ takes one of the following forms:*

$$f(x) = \left\lfloor \frac{x}{ab} \right\rfloor - \left\lfloor \frac{x}{b(a+b)} \right\rfloor - \left\lfloor \frac{x}{a(a+b)} \right\rfloor, \quad \text{where } \gcd(a, b) = 1, \tag{1.2}$$

$$f(x) = \left\lfloor \frac{x}{b(a-b)} \right\rfloor + \left\lfloor \frac{x}{2a(a-b)} \right\rfloor - \left\lfloor \frac{x}{2b(a-b)} \right\rfloor - \left\lfloor \frac{x}{a(a-b)} \right\rfloor - \left\lfloor \frac{x}{2ab} \right\rfloor, \tag{1.3}$$

where $\gcd(a, b) = \gcd(2, a-b) = 1$ *and* $a > b > 0$,

$$f(x) = \left\lfloor \frac{x}{(1/2)b(a-b)} \right\rfloor + \left\lfloor \frac{x}{a(a-b)} \right\rfloor - \left\lfloor \frac{x}{b(a-b)} \right\rfloor - \left\lfloor \frac{x}{(1/2)a(a-b)} \right\rfloor - \left\lfloor \frac{x}{ab} \right\rfloor, \tag{1.4}$$

where $\gcd(a, b) = \gcd(2, a) = \gcd(2, b) = 1$ *and* $a > b > 0$,

$$f(x) = \left\lfloor \frac{x}{b(a+b)} \right\rfloor + \left\lfloor \frac{x}{a(a+b)} \right\rfloor - \left\lfloor \frac{x}{2b(a+b)} \right\rfloor - \left\lfloor \frac{x}{2a(a+b)} \right\rfloor - \left\lfloor \frac{x}{2ab} \right\rfloor, \tag{1.5}$$

where $\gcd(a, b) = \gcd(2, a+b) = 1$,

$$f(x) = \left\lfloor \frac{x}{(1/2)b(a+b)} \right\rfloor + \left\lfloor \frac{x}{(1/2)a(a+b)} \right\rfloor - \left\lfloor \frac{x}{b(a+b)} \right\rfloor - \left\lfloor \frac{x}{a(a+b)} \right\rfloor - \left\lfloor \frac{x}{ab} \right\rfloor, \tag{1.6}$$

where $\gcd(a, b) = \gcd(2, a) = \gcd(2, b) = 1$;

(ii) *$f(x)$ is one of the 52 sporadic step functions given by*

$$(m_1, m_2, \ldots, m_N) = \left(\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_N} \right)$$

and

$$(m_{N+1}, m_{N+2}, \ldots, m_{2N+1}) = \left(\frac{M}{b_1}, \frac{M}{b_2}, \ldots, \frac{M}{b_{N+1}} \right)$$

for some $\mathbf{a}$ (*or permutation of* $\mathbf{a}$) *and* $\mathbf{b}$ (*or permutation of* $\mathbf{b}$) *listed in the second column of Table 2, where*

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_{N+1}).$$

This theorem is proved as a consequence of the equivalence of Problems 1.1 and 1.2, and the following theorem.

THEOREM 1.4. *Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_{K+1} n)!}$$

and suppose that $a_k \neq b_l$ *for all* k *and* l, *that* $\sum a_k = \sum b_l$, *and that*

$$\gcd(a_1, \ldots, a_K, b_1, \ldots, b_{K+1}) = 1.$$

Then $u_n(\mathbf{a}, \mathbf{b})$ *is an integer for all* n *if and only if either of the following statements hold:*

(i) $u_n = u_n(\mathbf{a}, \mathbf{b})$ *takes one of the following forms:*

$$u_n = \frac{[(a+b)n]!}{(an)!(bn)!} \quad \text{for } \gcd(a, b) = 1, \tag{1.7}$$

$$u_n = \frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!} \quad \text{for } \gcd(a, b) = 1 \text{ and } a > b, \tag{1.8}$$

$$u_n = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!} \quad \text{for } \gcd(a, b) = 1; \tag{1.9}$$

(ii) $(\mathbf{a}, \mathbf{b})$ *is one of the 52 sporadic parameter sets listed in the second column of Table 2.*

In Section 1.2 we note briefly that this theorem has immediate applications to the classification of cyclic quotient singularities.

It is a standard elementary exercise (see, for example, [**1**, Section 4.5]) to use the integrality of factorial ratios such as these to prove Chebyshev's elementary estimate for the prime counting function $\pi(x)$, namely, that there exist numbers c_1 and c_2, with $c_1 < 1 < c_2$, such that

$$c_1 \frac{x}{\log x} \leqslant \pi(x) \leqslant c_2 \frac{x}{\log x}$$

for all x large enough. It is interesting to note that the best such constants achievable from the factorial ratio sequences listed in Theorem 1.4 are those that were discovered by Chebyshev [**10**]. The factorial ratio sequence that gives these constants ($c_1 \approx 0.92$ and $c_2 \approx 1.11$) is

$$u_n = \frac{(30n)!n!}{(15n)!(10n)!(6n)!}, \tag{1.10}$$

which was in fact used by Chebyshev. (In truth, although Chebyshev's method is in some sense equivalent to using factorial ratios, it is not quite the same. Chebyshev actually makes more direct use of the fact that the corresponding step function

$$f(x) = \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x}{3} \right\rfloor - \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{30} \right\rfloor$$

takes on only the values 0 and 1.) The monodromy group associated with this u_n, that is, the monodromy group of the hypergeometric differential equation satisfied by $\mathbf{u}(\mathbf{a}, \mathbf{b})$, is $W(E_8)$, the Weyl group of the E_8 root system, which is the largest 'sporadic' finite primitive complex reflection group.

1.1. *Relation to the Nyman–Beurling formulation of the Riemann hypothesis*

Let

$$\rho_\alpha(x) = \left\lfloor \frac{\alpha}{x} \right\rfloor - \alpha \left\lfloor \frac{1}{x} \right\rfloor \in L^2(0,1).$$

There are various similar statements of the Nyman–Beurling formulation of the Riemann hypothesis. The original formulation is as follows.

THEOREM 1.5 (Nyman [**14**]). *The Riemann hypothesis is true if and only if*

$$\mathrm{span}_{L^2(0,1)}\{\rho_\alpha(x) \mid 0 < \alpha \leqslant 1\} = L^2(0,1).$$

REMARK 1.6. This appeared originally as Theorem 7 in [**14**]. A more general version, which says that linear combinations of these functions are dense in $L^p(0,1)$ if and only if $\zeta(\sigma+it)$ is zero-free for $\sigma > 1/p$, was given by Beurling [**7**]. For a proof of this theorem in the above form, using the theory of Hardy spaces, and more discussion, a good expository article is Balazard and Saias [**4**].

In fact, implicit in Beurling's proof is the fact that a sufficient condition for the Riemann hypothesis to be true is that the constant function can be approximated in $L^2(0,1)$ by linear combinations of functions of the form $\rho_{1/n}(x)$, for $n \in \mathbb{N}$. The converse was unknown for some time and is the following theorem of Báez-Duarte.

THEOREM 1.7 (Báez-Duarte [**2**]). *The Riemann hypothesis is true if and only if*

$$\chi_{(0,1)}(x) \in \mathrm{span}_{L^2(0,1)}\left\{\rho_\alpha(x) \;\middle|\; \frac{1}{\alpha} \in \mathbb{N}\right\},$$

where $\chi_{(0,1)}(x)$ is the characteristic function of the interval $(0,1)$.

REMARK 1.8. An alternative proof of this theorem is given by Burnol [**9**]. It is closely related to and motivated by the original, but it uses only pure complex analysis.

This was Vasyunin's motivation for studying Problem 1.1. Making the change of variables $x \to 1/x$, finite linear combinations of functions ρ_α become functions of the form

$$\sum_{k=1}^{n} c_k \left\lfloor \frac{x}{m_k} \right\rfloor, \quad m_k \in \mathbb{N},$$

such that

$$\sum_{k=1}^{n} \frac{c_k}{m_k} = 0,$$

and now the function space is $L^2((1,\infty), dx/x^2)$. Thus, the Nyman–Beurling criterion can be stated as: the Riemann hypothesis is equivalent to the existence of a sequence of functions $\{\phi_n(x)\}$, all of the above form such that $\phi_n(x) \to 1$ in $L^2((1,\infty), dx/x^2)$. Vasyunin studied families of functions that take only the values 0 and 1 to construct pointwise approximations to the constant function, and then explored the question stated in Problem 1.1. In particular, Vasyunin listed some infinite families of solutions to Problem 1.1 and 52 'sporadic' solutions, found by an extensive computer search, and he conjectured that his list was complete. Theorem 1.3 says that this is indeed the case. The step function corresponding to line 12 of Table 2 is not listed in [**17**]. However, Vasyunin stated that he found 21 seven-term step functions, but listed only 20, and so its omission must be a transcription error.

Báez-Duarte [**3**] has shown, as Vasyunin was well aware, that at least one of the pointwise approximations to the constant function that Vasyunin constructed diverges in $L^2((1,\infty), dx/x^2)$. It seems likely that the other sequences constructed by Vasyunin diverge as well. In light of this, it seems that, if this explicit approach is to give any insight into the Riemann hypothesis, it will be necessary to study step functions that take on more than two values. One natural question to ask might be: Can we classify all step functions of the form (1.1) that take only values in the set $\{0, 1, \ldots, D\}$ for fixed D? For $D > 1$, this question seems difficult. Using substantially different methods, a partial result on this question is obtained in a separate paper by Bell and the author [**5**].

1.2. *Application: cyclic quotient singularities*

Borisov [**8**] noted that there is a connection between integral factorial ratios and cyclic quotient singularities. In particular, we note that Borisov showed that Vasyunin's conjecture (Theorem 1.3) implies the following (see [**8**, Conjecture 1]).

Theorem 1.9. *Suppose that $d \geqslant 5$ and we have a one-parameter family of Gorenstein cyclic quotient singularities of dimension $2d+1$ with a Shokurov minimal log-discrepancy d. Then, up to the permutation of the coordinates in the $T^{(2d+1)}$, the corresponding points lie in the subtorus $x_1 + x_2 = 1$.*

For more on this connection, the reader should consult Borisov's paper.

1.3. *Notation*

Here $(\alpha)_n$ denotes the rising factorial

$$(\alpha)_n := (\alpha)(\alpha+1)(\alpha+2)\cdot\ldots\cdot(\alpha+n-1),$$

and, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$, the hypergeometric function ${}_nF_m(\alpha;\beta;z)$ is

$${}_nF_m(\alpha;\beta;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\cdot\ldots\cdot(\alpha_n)_k}{(\beta_1)_k(\beta_2)_k\cdot\ldots\cdot(\beta_m)_k}\frac{z^k}{k!}.$$

Also, $e(x) := \exp(2\pi i x) := e^{2\pi i x}$ and $\zeta_n = e(1/n)$ denotes the primitive nth root of unity with the smallest positive argument.

2. *Some preliminary definitions and notation*

Throughout this paper, unless otherwise specified, $\mathbf{a}$ and $\mathbf{b}$ denote ordered tuples of positive integers:

$$\mathbf{a} = (a_1, a_2, \ldots, a_K)$$

and

$$\mathbf{b} = (b_1, b_2, \ldots, b_L).$$

Also, $u_n(\mathbf{a},\mathbf{b})$ denotes the factorial ratio:

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1n)!(a_2n)!\cdot\ldots\cdot(a_Kn)!}{(b_1n)!(b_2n)!\cdot\ldots\cdot(b_Ln)!},$$

and $f(x;\mathbf{a},\mathbf{b})$ denotes the step function:

$$f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor .$$

Only later will we specify the requirement that $L = K + 1$.

It is useful to attach certain polynomials to $u_n(\mathbf{a},\mathbf{b})$ as follows.

Definition 2.1. Define $P(x) = P(\mathbf{a},\mathbf{b};x) \in \mathbb{Z}[x]$ and $Q(x) = Q(\mathbf{a},\mathbf{b};x) \in \mathbb{Z}[x]$ to be relatively prime polynomials such that

$$\frac{P(x)}{Q(x)} = \frac{(x^{a_1}-1)(x^{a_2}-1)\cdot \ldots \cdot (x^{a_K}-1)}{(x^{b_1}-1)(x^{b_2}-1)\cdot \ldots \cdot (x^{b_L}-1)}.$$

Then, for some $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_d$ and $\beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_d$, with $0 < \alpha_i, \beta_j \leqslant 1$, the polynomials P and Q factor in $\mathbb{C}[x]$ as

$$P(x) = (x - e(\alpha_1)) \cdot \ldots \cdot (x - e(\alpha_d))$$

and

$$Q(x) = (x - e(\beta_1)) \cdot \ldots \cdot (x - e(\beta_d)),$$

where $e(x) = e(2\pi i x)$.

Set $\alpha(\mathbf{a},\mathbf{b}) = \{\alpha_1, \ldots, \alpha_d\}$ and $\beta(\mathbf{a},\mathbf{b}) = \{\beta_1, \ldots, \beta_d\}$.

Remark 2.2. It is not too hard to see that, for $x \in [0,1]$, we have

$$f(x;\mathbf{a},\mathbf{b}) = \#\{\alpha_i \mid \alpha_i \leqslant x\} - \#\{\beta_i \mid \beta_i \leqslant x\}.$$

Thus, an alternative definition for $f(x;\mathbf{a},\mathbf{b})$ could be

$$f(x;\mathbf{a},\mathbf{b}) = \#(\alpha(\mathbf{a},\mathbf{b}) \cap [0,\{x\}]) - \#(\beta(\mathbf{a},\mathbf{b}) \cap [0,\{x\}]).$$

We occasionally make use of the notion of the interlacing of two sets, and so we state the following formally as a definition.

Definition 2.3 (Interlacing). We say that two finite sets of real numbers A and B interlace if the function

$$f(x) = \#\left((-\infty, x) \cap A\right) - \#\left((-\infty, x) \cap B\right)$$

either takes only the values 0 and 1, or takes only the values -1 and 0. In other words, there is an element of A in between any two elements of B, and an element of B in between any two elements of A.

We say that two sets A and B of complex numbers on the unit circle *interlace on the unit circle* if their arguments interlace on the real line, where we take the argument of a complex number to be in $[0, 2\pi)$.

3. *The connection between step functions and factorial ratios: the equivalence of Problems* 1.1 *and* 1.2

In this section we will prove Theorem 1.3 using Theorem 1.4 and the equivalence of Problems 1.1 and 1.2. We begin by stating a rather general theorem of Landau [**12**] that connects the integrality of factorial ratios with the non-negativity of related step functions.

THEOREM 3.1 (Landau [**12**]). *Let $a_{k,s}, b_{l,s} \in \mathbb{Z}_{\geqslant 0}$, for $1 \leqslant k \leqslant K, 1 \leqslant l \leqslant L$, and $1 \leqslant s \leqslant r$, and let*

$$A_k(x_1, x_2, \ldots, x_r) = \sum_{s=1}^{r} a_{k,s} x_s$$

and

$$B_l(x_1, x_2, \ldots, x_r) = \sum_{s=1}^{r} b_{l,s} x_s$$

(that is, A_k and B_l are linear forms in r variables with non-negative integral coefficients). Then the factorial ratio

$$\frac{\prod_{k=1}^{K} A_k(x_1, x_2, \ldots, x_r)!}{\prod_{l=1}^{L} B_l(x_1, x_2, \ldots, x_r)!}$$

is an integer for all $(x_1, \ldots, x_r) \in \mathbb{Z}^r_{\geqslant 0}$ if and only if the step function

$$F(y_1, \ldots, y_r) = \sum_{k=1}^{K} \lfloor A_k(y_1, \ldots, y_r) \rfloor - \sum_{l=1}^{L} \lfloor B_l(y_1, \ldots, y_r) \rfloor$$

is non-negative for all $(y_1, \ldots, y_r) \in [0, 1]^r$.

Proof. See [**12**] for the proof of this theorem. □

The special case of this theorem that we use is the following.

LEMMA 3.2. *Let*

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_L n)!}.$$

Then u_n is an integer for all n if and only if the function

$$f(x) = f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

is non-negative for all x between 0 and 1.

Proof. Take $A_k(x) = a_k x$ and $B_l(x) = b_l x$ in Theorem 3.1. □

It also turns out that, if $f(x; \mathbf{a}, \mathbf{b})$ is ever negative, then every prime that is large enough occurs as a factor in the denominator of $u_n(\mathbf{a}, \mathbf{b})$ for some n.

LEMMA 3.3. *Let*

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_L n)!}.$$

If u_n is not an integer for some n, then there exists some integer P such that, for each prime $p > P$, there exists some n such that $v_p(u_n) < 0$ (where $v_p(u_n)$ is the p-adic valuation of u_n).

Proof. Consider the p-adic valuation of $n!$. We have

$$v_p(n!) = \sum_{\alpha=1}^{\infty} \left\lfloor \frac{n}{p^\alpha} \right\rfloor .$$

Thus we have

$$v_p(u_n) = \sum_{\alpha=1}^{\infty} f\left(\frac{n}{p^\alpha}\right),$$

where $f(x) = f(x; \mathbf{a}, \mathbf{b})$.

Assuming that u_n is not always an integer, we know from Lemma 3.2 that $f(x)$ is negative for some x. Since f is a step function, it follows that there is some interval, say $[\beta, \beta+\epsilon]$, such that $f(x) < 0$ for all $x \in [\beta, \beta+\epsilon]$. Additionally, we know that there is some $\delta > 0$ such that $f(x) = 0$ for all $x \in [0, \delta]$. If we could find some n and p such that $n/p \in [\beta, \beta+\epsilon]$ and $n/p^2 \in [0, \delta]$, then we would have $f(n/p) < 0$ and $f(n/p^\alpha) = 0$ for all $\alpha > 1$, and so we would clearly have $\nu_p(u_n) < 0$.

Now, such an n and p need to simultaneously satisfy the two inequalities

$$p\beta \leqslant n \leqslant p(\beta + \epsilon)$$

and

$$0 \leqslant n \leqslant p^2\delta.$$

For p large enough, say $p > P_1$, we have $p^2\delta > p(\beta+\epsilon)$, and so it is sufficient for n and p to satisfy the first of the inequalities. Moreover, for any p large enough, say $p > P_2$, we have $p\epsilon > 1$, so that there will in fact be an integer n between $p\beta$ and $p(\beta+\epsilon)$. Therefore, in fact, for any $p > P = \max(P_1, P_2)$, we have that there exists an n such that $\nu_p(u_n) < 0$. $\square$

Along with Lemma 3.2, the following lemma, which is a simple generalization of Proposition 3 in [**17**] will yield the full equivalence of Problems 1.1 and 1.2.

LEMMA 3.4. *Suppose that $f(x)$ is a function of the form*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor ,$$

with $a_k, b_l \in \mathbb{Z}$, and that $f(x)$ is bounded for all $x \in \mathbb{R}$. Then $\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$ and, for any n, there exists some x such that $f(x) = -n$ if and only if there exists some x' such that $f(x') = L - K + n$. In particular, $f(x)$ is non-negative if and only if the maximum value of f is $L - K$.

Proof. The first assertion is clear, as $f(n) = n\left(\sum a_k - \sum b_j\right)$ for $n \in \mathbb{Z}$, and so, if $\sum a_k \neq \sum b_l$, then $f(x)$ is unbounded. Now we know that $f(x)$ is periodic with period 1. Now, for any z that is not an integer, we have $\lfloor z \rfloor + \lfloor -z \rfloor = -1$, and so, for any z for which none of $a_i z$ and $b_j z$ is an integer, we have

$$f(z) + f(-z) = L - K,$$

from which the assertion follows. $\square$

The following lemma describes explicitly the equivalence between Problems 1.1 and 1.2, in a slightly more general form.

LEMMA 3.5. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_K)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_L)$*, and let*

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_L).$$

Set

$$(m_1, m_2, \ldots, m_{L+K}) = \left(\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_L}, \frac{M}{b_1}, \frac{M}{b_2}, \ldots, \frac{M}{b_k}\right).$$

Then the following statements are equivalent.

(i) *The function*

$$f(x) = \sum_{i=1}^{K} \left\lfloor \frac{x}{m_i} \right\rfloor - \sum_{i=K+1}^{K+L} \left\lfloor \frac{x}{m_i} \right\rfloor$$

takes on values only in the range $0, 1, \ldots, L - K$.

(ii) *We have* $\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$ *and*

$$u_n = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_L n)!}$$

is an integer for all $n \in \mathbb{N}$.

Moreover, if $L - K = 1$*, we may add the following statement.*

(iii) *We have that* $\alpha(\mathbf{a}, \mathbf{b})$ *and* $\beta(\mathbf{a}, \mathbf{b})$ *interlace.*

Proof. Here $f(x)$ differs from $f(x; \mathbf{a}, \mathbf{b})$ only by a change of variables, and so Proposition 3.2 tells us that $u_n \in \mathbb{Z}$ for all n if and only if $f(x) > 0$ for all $x \in [0, 1]$. Additionally, the boundedness of $f(x)$ is equivalent to the statement that $\sum a_k = \sum b_l$, and Lemma 3.4 tells us that $f(x)$ is bounded and non-negative if and only if its maximum value is $L - K$.

Part (iii) is simply an issue of terminology, and follows directly from the alternative definition of $f(x; \mathbf{a}, \mathbf{b})$ given in Remark 2.2. $\square$

Using this equivalence, we can prove Theorem 1.3 as an application of Theorem 1.4.

Proof of Theorem 1.3 (*Using Theorem* 1.4). The only complication that remains is that of classifying solutions with a greatest common divisor 1. Consider the map $\phi : \mathbb{N}^K \times N^L \to \mathbb{N}^K \times \mathbb{N}^L$ given by

$$\phi(a_1, a_2, \ldots, a_K, b_1, \ldots, b_L) = \left(\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_K}, \frac{M}{b_1}, \ldots, \frac{M}{b_L}\right),$$

where

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_K, b_1, \ldots, b_L).$$

The image of ϕ is all $(K + L)$-tuples with a greatest common divisor 1 and ϕ is bijective on this subset. Thus ϕ, in combination with Lemma 3.5, gives a bijection between the solutions to Problem 1.1 with a greatest common divisor 1 and the solutions to Problem 1.2 with a greatest common divisor 1.

When we apply this map to the three families of solutions to Problem 1.2 listed in Theorem 1.3, we get the five families of solutions to Problem 1.1 listed in Theorem 1.3, and the 52 sporadic solutions are given by the 52 sporadic solutions to Problem 1.2. $\square$

4. *The connection between factorial ratios and hypergeometric series*

Rodriguez-Villegas [15] observed a connection between hypergeometric series and factorial ratio sequences. The purpose of this section is to write out this connection explicitly in order to use it for our classification.

We begin with a lemma to show that the generating function for $u_n(\mathbf{a},\mathbf{b})$ is in fact a hypergeometric series.

LEMMA 4.1. *Let*

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1n)!(a_2n)!\cdot\ldots\cdot(a_Kn)!}{(b_1n)!(b_2n)!\cdot\ldots\cdot(b_Ln)!}$$

and

$$\mathbf{u}(\mathbf{a},\mathbf{b};z) = \sum_{n=0}^{\infty} u_n(\mathbf{a},\mathbf{b})z^n.$$

Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_d) = \alpha(\mathbf{a},\mathbf{b})$ *and* $\beta = (\beta_1,\beta_2,\ldots,\beta_d) = \beta(\mathbf{a},\mathbf{b})$, *as in Definition* 2.1, *and let*

$$C = \frac{a_1^{a_1}\cdot\ldots\cdot a_K^{a_K}}{b_1^{b_1}\cdot\ldots\cdot b_L^{b_L}}.$$

If $L > K$, *then* $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ *is the hypergeometric series*

$$\mathbf{u}(\mathbf{a},\mathbf{b};z) = {}_dF_{d-1}\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_d\\ \beta_1,\beta_2,\ldots,\beta_{d-1}\end{matrix};Cz\right).$$

Otherwise, if $L \leqslant K$, *then* $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ *is the hypergeometric series*

$$\mathbf{u}(\mathbf{a},\mathbf{b};z) = {}_{d+1}F_d\left(\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_d,1\\ \beta_1,\beta_2,\ldots,\beta_d\end{matrix};Cz\right).$$

Proof. Examine the ratio between two consecutive terms

$$\begin{aligned}A(n+1) &= \frac{u_{n+1}(\mathbf{a},\mathbf{b})}{u_n(\mathbf{a},\mathbf{b})}\\ &= \frac{(a_1(n+1))!(a_2(n+1))!\cdot\ldots\cdot(a_K(n+1))!}{(b_1(n+1))!(b_2(n+1))!\cdot\ldots\cdot(b_L(n+1))!}\times\left[\frac{(a_1n)!(a_2n)!\cdot\ldots\cdot(a_Kn)!}{(b_1n)!(b_2n)!\cdot\ldots\cdot(b_Ln)!}\right]^{-1}.\end{aligned}$$

After cancellation, this can be written as

$$A(n+1) = \frac{(a_1n+1)(a_1n+2)\cdot\ldots\cdot(a_1n+a_1)(a_2n+1)\cdot\ldots\cdot(a_Kn+a_K)}{(b_1n+1)(b_1n+2)\cdot\ldots\cdot(b_1n+b_1)(b_2n+1)\cdot\ldots\cdot(b_Ln+b_L)}.$$

Now, if we factor out the coefficients of n in each term, then we get

$$A(n+1) = C\frac{(n+1/a_1)(n+2/a_1)\cdot\ldots\cdot(n+a_1/a_1)(n+1/a_2)\cdot\ldots\cdot(n+a_K/a_K)}{(n+1/b_1)(n+2/b_1)\cdot\ldots\cdot(n+b_1/b_1)(n+1/b_2)\cdot\ldots\cdot(n+b_L/b_L)},$$

where

$$C = \frac{a_1^{a_1}\cdot\ldots\cdot a_K^{a_K}}{b_1^{b_1}\cdot\ldots\cdot b_L^{b_L}}.$$

If we remove the common factors in the fraction, then, for exactly the same α and β as in Definition 2.1, we have

$$A(n+1) = C\frac{(n+\alpha_1)\cdot\ldots\cdot(n+\alpha_d)}{(n+\beta_1)\cdot\ldots\cdot(n+\beta_d)}.$$

Now, $u_0(\mathbf{a},\mathbf{b}) = 1$, and so we have in general

$$u_n(\mathbf{a},\mathbf{b}) = \prod_{k=1}^{n} A(k) = C^n \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_d)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_d)_n}.$$

Now, if $L > K$, then $\beta_d = 1$, and so

$$u_n(\mathbf{a},\mathbf{b}) = \frac{C^n}{n!} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_d)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_{d-1})_n}$$

and

$$\mathbf{u}(\mathbf{a},\mathbf{b};z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_d)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_{d-1})_n} \frac{(Cz)^n}{n!} = {}_dF_{d-1}\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_d \\ \beta_1, \beta_2, \ldots, \beta_{d-1} \end{matrix}; Cz \right).$$

If, on the other hand, $L \leqslant K$, then $\beta_d \neq 1$, and so we instead write

$$u_n(\mathbf{a},\mathbf{b}) = \frac{C^n}{n!} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_d)_n (1)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_d)_n},$$

and we find that

$$\mathbf{u}(\mathbf{a},\mathbf{b};z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdot \ldots \cdot (\alpha_d)_n (1)_n}{(\beta_1)_n \cdot \ldots \cdot (\beta_d)_n} \frac{(Cz)^n}{n!} = {}_{d+1}F_d\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_d, 1 \\ \beta_1, \beta_2, \ldots, \beta_d \end{matrix}; Cz \right).$$

□

EXAMPLE 4.2. Let $\mathbf{a} = (30, 1)$ and $\mathbf{b} = (15, 10, 6)$, and

$$u_n = u_n(\mathbf{a},\mathbf{b}) = \frac{(30n)!n!}{(15n)!(10n)!(6n)!}.$$

Consider the ratio u_{n+1}/u_n. This is

$$\frac{(30n+1)(30n+2) \cdot \ldots \cdot (30n+30)(n+1)}{(15n+1) \cdot \ldots \cdot (15n+15)(10n+1) \cdot \ldots \cdot (10n+10)(6n+1) \cdot \ldots \cdot (6n+6)}.$$

Factoring out the coefficients of n in each term in the products, we get

$$\frac{30^{30}(n+1/30)(n+2/30) \cdot \ldots \cdot (n+30/30)(n+1)}{15^{15}10^{10}6^6(n+1/15) \cdot \ldots \cdot (n+15/15)(n+1/10) \cdot \ldots \cdot (n+10/10)(n+1/6) \cdot \ldots \cdot (n+6/6)}.$$

Now there is a lot of clear cancellation in the fraction, and we see that this is

$$\frac{\begin{matrix} 30^{30}(n+1/30)(n+7/30)(n+11/30)(n+13/30)(n+17/30) \\ (n+19/30)(n+23/30)(n+29/30) \end{matrix}}{15^{15}10^{10}6^6(n+1/5)(n+1/3)(n+2/5)(n+1/2)(n+3/5)(n+2/3)(n+4/5)(n+1)},$$

which tells us that

$$\sum_{n\geqslant 1} u_n z^n = {}_8F_7\left(\tfrac{1}{30}, \tfrac{7}{30}, \tfrac{11}{30}, \tfrac{13}{30}, \tfrac{17}{30}, \tfrac{19}{30}, \tfrac{23}{30}, \tfrac{29}{30}; \tfrac{1}{5}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{4}{5}; Cz\right),$$

where

$$C = \frac{30^{30}}{15^{15}10^{10}6^6}.$$

We need to know that the *hypergeometric series* attached to a factorial ratio is essentially unique. We prove this in the next two lemmas.

LEMMA 4.3. *Suppose that* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_K > 0$ *and* $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_L > 0$, *and that*

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_L n)!} = 1$$

for all $n \geqslant 1$. *Then* $K = L$ *and* $\mathbf{a} = \mathbf{b}$.

Proof. The cases $K \geqslant L$ and $K \leqslant L$ are symmetric, and so we may as well assume that $K \leqslant L$. We prove the case $K = 1$ and then proceed by induction on K.

If $K = 1$ and $a_1 < b_1$, then it is clear that $u_n \to 0$ as $n \to \infty$. On the other hand, if $a_1 > b_1$, then, by Dirichlet's theorem on primes in arithmetic progressions, there exists some $m > 1$ such that $a_1 m - 1 = p$ is prime. Then p divides the numerator of u_m but not the denominator, and so $u_m \neq 1$. Now, if $a_1 = b_1$, then it is clear that $L = K = 1$.

The case for general K proceeds similarly. We need only show that $a_1 = b_1$, and we are finished by induction. Again, if $a_1 > b_1$, then there is some $m > 1$ such that $a_1 m - 1 = p$ is prime, and p divides the numerator of u_m but not the denominator. If, on the other hand, $b_1 > a_1$, then we just reverse the argument and find an m and p such that p divides the denominator of u_m but not the numerator. $\square$

LEMMA 4.4. *The map*

$$(\mathbf{a}, \mathbf{b}) \longrightarrow u_n(\mathbf{a}, \mathbf{b}; z)$$

is one-to-one on the set of pairs $(\mathbf{a}, \mathbf{b})$ *such that* $a_k \neq b_l$ *for all* k *and* l, *and* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_K$ *and* $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_L$.

Proof. For some $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$, we have

$$u_n(\mathbf{a}, \mathbf{b}; z) = u_n(\mathbf{a}', \mathbf{b}'; z)$$

if and only if

$$u_n(\mathbf{a}, \mathbf{b}) = u_n(\mathbf{a}', \mathbf{b}')$$

for all n. In this case, we can rewrite this as

$$u_n(\mathbf{a}, \mathbf{b})(u_n(\mathbf{a}', \mathbf{b}'))^{-1} = u_n(\mathbf{a} \cup \mathbf{b}', \mathbf{b} \cup \mathbf{a}') = 1.$$

Now it follows from Lemma 4.3 that $\mathbf{a} \cup \mathbf{b}'$ is a permutation of $\mathbf{b} \cup \mathbf{a}'$. Thus $\mathbf{a}'$ is a permutation of $\mathbf{a}$ and $\mathbf{b}'$ is a permutation of $\mathbf{b}$. $\square$

REMARK 4.5. It is also possible to state Lemma 4.4 in an algorithmic manner. Roughly speaking, given parameters $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_{d-1})$ that come from a factorial ratio, we can form the polynomials $P(x)$ and $Q(x)$ from Definition 2.1. It is then possible to add extra factors to $P(x)$ and $Q(x)$ together to obtain

$$\frac{P(x)}{Q(x)} = \frac{(x^{a_1} - 1)(x^{a_2} - 1) \cdot \ldots \cdot (x^{a_K} - 1)}{(x^{b_1} - 1)(x^{b_2} - 1) \cdot \ldots \cdot (x^{b_L} - 1)},$$

and to recover $\mathbf{a}$ and $\mathbf{b}$. In this manner, if we did not already know about the 52 sporadic integer factorial ratio sequences from Vasyunin's work, we could recover them from the work of Beukers and Heckman [**6**] described in Section 5.1.

The main interest in looking at hypergeometric series attached to factorial ratio sequences comes from the following observation of Rodriguez-Villegas [**15**].

THEOREM 4.6 (Rodriguez-Villegas [**15**]). *Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdot \ldots \cdot (a_K n)!}{(b_1 n)!(b_2 n)! \cdot \ldots \cdot (b_L n)!},$$

with $\sum_{K=1}^{K} a_k = \sum_{l=1}^{L} b_l$, *and let*

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n.$$

Then $\mathbf{u}(\mathbf{a}, \mathbf{b}; z)$ *is algebraic over* $\mathbb{Q}(t)$ *if and only if* $L - K = 1$ *and* $u_n(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}$ *for all* $n \geqslant 0$.

Proof of Theorem 4.6 (*Part 1*). We begin by proving that, if the generating function is algebraic, then u_n is in fact integral. In particular, it follows from Lemmas 3.2 and 3.4 that this will imply that we must have $L - K \geqslant 1$, which is, in fact, all that we need from this part of the proof.

A theorem of Eisenstein [**11**] asserts that, if $\mathbf{u}(\mathbf{a}; \mathbf{b}; z)$ is algebraic, then there exists an N such that $u_n(\mathbf{a}, \mathbf{b}) \cdot N^n$ is an integer for all n. But Lemma 3.3 implies that the set of primes occurring in the denominator of some $u_n(\mathbf{a}, \mathbf{b})$ is either empty or infinite. So, if such an N exists, then we are able to take $N = 1$, which implies that $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all n. □

The remainder of this proof relies on Landau's theorem and the following lemma of Beukers and Heckman [**6**].

Lemma 4.7 (Beukers and Heckman [**6**]). *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and* $\beta_1, \beta_2, \ldots, \beta_{n-1}$ *be rational numbers with a common denominator* M. *The hypergeometric function*

$${}_nF_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_{n-1}; z)$$

is algebraic if and only if, for all k *relatively prime to* M, *the sequences*

$$e(k\alpha_1), \ldots, e(k\alpha_n)$$

and

$$e(k\beta_1), \ldots, e(k\beta_{n-1}), 1$$

interlace on the unit circle.

Proof. This follows from [**6**, Theorem 4.8] and the fact that this function is algebraic if and only if its monodromy group is finite. □

For the case of the hypergeometric functions that are generating series for $u_n(\mathbf{a}, \mathbf{b})$, we can make this lemma slightly stronger.

Lemma 4.8. *The series*

$$\mathbf{u}(\mathbf{a}; \mathbf{b}; z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n$$

is an algebraic function if and only if $\alpha = \alpha(\mathbf{a}, \mathbf{b})$ *and* $\beta = \beta(\mathbf{a}, \mathbf{b})$ *interlace on* $[0, 1]$.

Proof. In our case the α_i and β_j are rational numbers in $(0, 1]$. Suppose that they have a common denominator M. Recall that the numbers $e(\alpha_i)$ are roots of the polynomial $P(\mathbf{a}, \mathbf{b}; x)$, and that P is the product of cyclotomic polynomials, say $P = \Phi_{m_1}\Phi_{m_2} \cdot \ldots \cdot \Phi_{m_l}$. Then for any $(k, M) = 1$ we also have $(k, m_i) = 1$ for all m_i. Therefore the map $\alpha \to \alpha^k$ simply permutes the roots of any Φ_{m_i}. In particular, it permutes the roots of P, and hence permutes the numbers $e(\alpha_i)$.

The exact same argument applies for β. Thus we have that α and β interlace on $[0,1]$ if and only if $e(\alpha_i)^k$ and $e(\beta_j)^k$ interlace on the unit circle for all k with $(k,M)=1$.

In particular, $\mathbf{u}(\mathbf{a};\mathbf{b};z)$ is algebraic if and only if α and β interlace on $[0,1]$. □

Combining these lemmas finishes the proof of Theorem 4.6.

Proof of Theorem 4.6 (*Part 2*). Suppose that $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ is algebraic. Then we know that $L-K \geqslant 1$, from the first part of the proof. Note that the number of copies of the number 1 in the set β is $L-K$. However, if α and β are to interlace, then no value can be repeated, and so we must have $L-K=1$.

Now, if $L-K=1$, then from Lemma 3.5 we know that $u_n(\mathbf{a},\mathbf{b})$ is integral if and only if α and β interlace. From Lemma 4.8 we know that this is equivalent to $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ being algebraic. □

5. *A classification of integral factorial ratios*

5.1. *Monodromy for hypergeometric functions* ${}_nF_{n-1}$

This section is an application of the work of Beukers and Heckman [6], and so we begin by restating a few necessary theorems and definitions.

Definition 5.1 (Hypergeometric groups). Let $w_1,\ldots,w_n$ and $z_1,\ldots,z_n$ be complex numbers with $w_i \neq z_j$ for all i and j. The *hypergeometric group* $H(\mathbf{w},\mathbf{z})$ with numerator parameters $w_1,\ldots,w_n$ and denominator parameters $z_1,\ldots,z_n$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by the elements

$$h_0, h_1, \text{ and } h_\infty$$

such that

$$h_0 h_1 h_\infty = 1$$

and

$$\det(t-h_\infty)=\prod_{j=1}^{n}(t-w_j),$$

$$\det(t-h_0^{-1})=\prod_{j=1}^{n}(t-z_j),$$

and such that h_1-1 has a rank 1.

Hypergeometric groups are precisely those groups that occur as monodromy groups for hypergeometric functions. Specifically, we have the following.

Proposition 5.2. *The monodromy group for the hypergeometric function*

$${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};z)$$

is a hypergeometric group with numerator parameters

$$e(\alpha_1), e(\alpha_2),\ldots,e(\alpha_n)$$

and denominator parameters

$$e(\beta_1), e(\beta_2),\ldots,e(\beta_{n-1}), 1.$$

Proof. This is Proposition 3.2 in [6]. □

In categorizing hypergeometric groups, it is useful to consider the following special subgroup.

DEFINITION 5.3. The subgroup $H_r(\mathbf{w},\mathbf{z})$ of $H(\mathbf{w},\mathbf{z})$ generated by $h_\infty^k h_1 h_\infty^{-k}$ for $k \in \mathbb{Z}$ is called the reflection subgroup of $H(\mathbf{w},\mathbf{z})$.

Also, the classification of hypergeometric groups splits into a primitive case and an imprimitive case, as in the following definition.

DEFINITION 5.4. Let $G \subset \mathrm{GL}(V)$ be a subgroup acting irreducibly on a complex vector space V. Then G is called *imprimitive* if there exists a direct sum decomposition $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$, with $n > 1$ and $\dim V_i > 0$ for all i, such that the action of G on V permutes the subspaces V_i. Otherwise, G is called *primitive*.

The existence of the following two theorems explains some of the usefulness of considering the reflection subgroup of a hypergeometric group and in considering when a hypergeometric group is primitive.

THEOREM 5.5. *The reflection subgroup $H_r(\mathbf{w},\mathbf{z})$ of $H(\mathbf{w},\mathbf{z})$ acts reducibly on $\mathbb{C}^n$ if and only if there exists a root of unity $\zeta \neq 1$ such that multiplication by ζ permutes both the elements of $\mathbf{w}$ and the elements of $\mathbf{z}$. Moreover, if $H_r(\mathbf{w},\mathbf{z})$ is reducible, then $H(\mathbf{w},\mathbf{z})$ is imprimitive.*

Proof. This is Theorem 5.3 in [6]. □

THEOREM 5.6. *Suppose that $H_r(\mathbf{w},\mathbf{z})$ is irreducible. Then H is imprimitive if and only if there exist $p, q \in N$ with $p+q=n$ and $(p,q)=1$, and $A, B, C \in \mathbb{C}^*$ such that $A^n = B^p C^q$ and such that*

$$\{w_1, \ldots, w_n\} = \{A, A\zeta_n, A\zeta_n^2, \ldots, A\zeta_n^{n-1}\}$$

and

$$\{z_1, \ldots, z_n\} = \{B, B\zeta_p, B\zeta_p^2, \ldots, B\zeta_p^{p-1}, C, C\zeta_q, C\zeta_q^2, \ldots, C\zeta_q^{q-1}\},$$

where $\zeta_n = e(1/n)$, or with the same sets of equalities with $\mathbf{w}$ and $\mathbf{z}$ reversed.

Proof. This is Theorem 5.8 in [6]. □

As defined, hypergeometric groups are subgroups of $\mathrm{GL}_n(\mathbb{C})$. The following proposition tells us when a hypergeometric group is defined over $\mathrm{GL}_n(R)$ for $R \subset \mathbb{C}$.

PROPOSITION 5.7. *Suppose that $w_1, \ldots, w_n, z_1, \ldots, z_n \in \mathbb{C}^*$ with $w_i \neq z_j$ for all i and j. Let $A_1, \ldots A_n, B_1, \ldots, B_n$ be defined by*

$$\prod_{j=1}^{n} (t - w_j) = t^n + A_1 t^{n-1} + \ldots + A_n,$$

and

$$\prod_{j=1}^{n}(t-z_j) = t^n + B_1 t^{n-1} + \ldots + B_n.$$

Then, relative to a suitable basis, the hypergeometric group $H(\mathbf{w},\mathbf{z})$ is defined over the ring $\mathbb{Z}[A_1,\ldots,A_n,B_1,\ldots,B_n,A_n^{-1},B_n^{-1}]$.

Proof. This is Corollary 3.6 in [**6**], and follows directly from a theorem of Levelt [**13**, Theorem 1.1]. □

We need to state one more definition, and then we will be ready to state the main classification theorem of Beukers and Heckman that we are interested in.

DEFINITION 5.8. A scalar shift of the hypergeometric group $H(\mathbf{w},\mathbf{z})$ is a hypergeometric group $H(d\mathbf{w},d\mathbf{z}) = H(dw_1,dw_2,\ldots,dw_n;dz_1,dz_2,\ldots,dz_n)$ for some $d \in C^*$.

Our main interest in the work of Beukers and Heckman comes from the following theorem.

THEOREM 5.9. *Let $n \geqslant 3$ and let $H(\mathbf{w},\mathbf{z}) \subset \mathrm{GL}_n(\mathbb{C})$ be a primitive hypergeometric group whose parameters are roots of unity and generate the field $\mathbb{Q}(\zeta_h)$. Then $H(\mathbf{w},\mathbf{z})$ is finite if and only if, up to a scalar shift, the parameters have the form $w_1^k,w_2^k,\ldots,w_n^k;z_1^k,z_2^k,\ldots,z_n^k$, where $\gcd(h,k)=1$ and the exponents of either $w_1,\ldots,w_n;z_1,\ldots,z_n$ or $z_1,\ldots,z_n;w_1,\ldots,w_n$ are listed in Table 8.3 in* [**6**].

Proof. This is Theorem 7.1 in [**6**]. □

5.2. *The classification*

From now on, we set $L = K+1$. We are interested in ratios where the parameters have a greatest common divisor 1. The following lemma shows that this condition translates nicely into the reflection group of the monodromy group being irreducible.

LEMMA 5.10. *Let $u_n = u_n(\mathbf{a},\mathbf{b})$ and $\mathbf{u} = \mathbf{u}(\mathbf{a},\mathbf{b};z)$. Let $H(\mathbf{u})$ be the hypergeometric group associated to $\mathbf{u}$ and let $H_r(\mathbf{u})$ be the reflection subgroup of $H(\mathbf{u})$. Suppose that u_n is an integer for all n. Then $H_r(\mathbf{u})$ acts reducibly on $\mathbb{C}$ if and only if*

$$\gcd(a_1,a_2,\ldots,a_K,b_1,b_2,\ldots,b_{K+1}) > 1.$$

Moreover, if $H_r(\mathbf{u})$ is reducible, then $H(\mathbf{u})$ is imprimitive.

Proof. Let $P = P(\mathbf{a},\mathbf{b};x)$ and $Q = Q(\mathbf{a},\mathbf{b};x)$. Then we have

$$\frac{P}{Q} = \frac{(x-e(\alpha_1))\cdot\ldots\cdot(x-e(\alpha_d))}{(x-e(\beta_1))\cdot\ldots\cdot(x-e(\beta_d))} = \frac{(x^{a_1}-1)\cdot\ldots\cdot(x^{a_K}-1)}{(x^{b_1}-1)\cdot\ldots\cdot(x^{b_{K+1}}-1)},$$

and $H(\mathbf{u})$ is a hypergeometric group with numerator parameters

$$e(\alpha_1),e(\alpha_2),\ldots,e(\alpha_d)$$

and denominator parameters

$$e(\beta_1),e(\beta_2),\ldots,e(\beta_d).$$

From Theorem 5.5 we know that $H_r(\mathbf{u})$ acts reducibly on $\mathbb{C}$ if and only if there exists some $\gamma \not\equiv 0 \pmod 1$ such that

$$\{e(\alpha_1), e(\alpha_2), \ldots, e(\alpha_d)\} = \{e(\alpha_1+\gamma), e(\alpha_2+\gamma), \ldots, e(\alpha_d+\gamma)\}$$

and

$$\{e(\beta_1), e(\beta_2), \ldots, e(\beta_d)\} = \{e(\beta_1+\gamma), e(\beta_2+\gamma), \ldots, e(\beta_d+\gamma)\}.$$

In this case, $e(\gamma)$ is necessarily a primitive Mth root of unity for some M and, by raising it to an appropriate power, we may assume that γ is a primitive pth root of unity for some prime p. We show that, if multiplication by $e(\gamma)$ gives the desired permutations, then p divides all of the a_k and all of the b_j.

Recall that P and Q are products of cyclotomic polynomials, and let Z_M denote the set of primitive Mth roots of unity. Notice that, if $(M,p)=1$, then multiplication by $e(\gamma)$ maps all primitive Mth roots of unity to primitive Mpth roots of unity, while some primitive Mpth roots of unity are mapped to others, and some are mapped to primitive pth roots of unity. Thus, multiplication by $e(\gamma)$ gives a permutation of $Z_M \cup Z_{Mp}$. On the other hand, multiplication by $e(\gamma)$ simply permutes each of the sets Z_{Mp^e} for $e>1$.

Thus, if multiplication by $e(\gamma)$ separately permutes the α_i and the β_j, then, whenever P or Q has a factor Φ_M with $(M,p)=1$, it must also have a factor Φ_{Mp}. Suppose then that there are some a_l or b_k coprime to p, and assume that M is the largest of these. Then (x^M-1) would have Φ_M as a factor, and any other term $(x^{a_k}-1)$ or $(x^{b_j}-1)$ that had Φ_M as a factor would necessarily have Φ_{Mp} as a factor as well. Ultimately, one of the polynomials P or Q would not have the terms Φ_M and Φ_{pM} properly paired as factors. Thus, if p does not divide $\gcd(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$, then multiplication by a primitive pth root of unity does not separately permute $\alpha(\mathbf{a}, \mathbf{b})$ and $\beta(\mathbf{a}, \mathbf{b})$.

On the other hand, if p does divide $\gcd(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$, then multiplication by $e(1/p)$ permutes the roots of each of the factors $(x^{a_k}-1)$ and $(x^{b_l}-1)$, and hence it separately permutes the roots of P and the roots of Q. □

Vasyunin noticed that the step functions corresponding to

$$u_n = \frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!} \quad \text{with } b<a \tag{5.1}$$

and

$$u_n = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!} \tag{5.2}$$

are non-negative. Thus, for both of these families, u_n is an integer for all n. It turns out that, when a and b are not both odd, these two infinite families give exactly those with factorial ratios with gcd 1, for which the hypergeometric group is imprimitive. On the other hand, when a and b are both odd, these come from scalar shifts of the hypergeometric groups associated to binomial coefficients.

LEMMA 5.11. *Let $u_n = u_n(\mathbf{a}, \mathbf{b})$ and $\mathbf{u} = \mathbf{u}(\mathbf{a}, \mathbf{b}; z)$. Let $H(\mathbf{u})$ be the hypergeometric group associated to $\mathbf{u}$ and let $H_r(\mathbf{u})$ be the reflection subgroup of $H(\mathbf{u})$. Suppose that u_n is an integer for all n and that $H_r(\mathbf{u})$ is irreducible. Then $H(\mathbf{u})$ is imprimitive if and only if u_n is of the form (5.1) or (5.2), with a and b not both odd.*

Proof. Again let $P = P(\mathbf{a}, \mathbf{b}; x)$ and $Q = Q(\mathbf{a}, \mathbf{b}; x)$. Suppose that u_n is of the form (5.1). Then we have

$$\frac{P}{Q} = \frac{(x^{2a}-1)(x^b-1)}{(x^{2b}-1)(x^a-1)(x^{a-b}-1)} = \frac{(x^a+1)}{(x^b+1)(x^{a-b}-1)},$$

which is in the lowest terms if a and b are not both odd. Then the numerator parameters for $H(\mathbf{u})$ are

$$A, A\zeta_a^1, A\zeta_a^2, \ldots, A\zeta_a^{a-1}$$

and the denominator parameters are

$$B, B\zeta_b, B\zeta_b^2, \ldots, B\zeta_b^{b-1}, \zeta_{a-b}, \zeta_{a-b}^2, \ldots, \zeta_{a-b}^{a-b-1}, 1,$$

where $A = \zeta_{2a}$ and $B = \zeta_{2b}$ satisfy $A^a = B^b = -1$, and so these parameters satisfy the condition of Theorem 5.6, and $H(\mathbf{u})$ is imprimitive.

Similarly, if u_n is of the form (5.2), then we have

$$\frac{P}{Q} = \frac{(x^{2a}-1)(x^{2b}-1)}{(x^a-1)(x^b-1)(x^{a+b}-1)} = \frac{(x^a+1)(x^b+1)}{x^{a+b}-1},$$

again in the lowest terms if a and b are not both odd, and so $H(\mathbf{u})$ is a hypergeometric group with numerator parameters

$$A, A\zeta_a^1, A\zeta_a^2, \ldots, A\zeta_a^{a-1}, B, B\zeta_b, B\zeta_b^2, \ldots, B\zeta_b^{b-1}$$

and denominator parameters

$$\zeta_{a+b}, \zeta_{a+b}^2, \ldots, \zeta_{a+b}^{a+b},$$

where $A = \zeta_{2a}$ and $B = \zeta_{2b}$. Now A and B satisfy $A^a = B^b$, and so $H(\mathbf{u})$ again satisfies the conditions of the theorem.

To see the converse, suppose that the numerator parameters of $H(u)$ are of the form

$$A, A\zeta_a, A\zeta_a^2, \ldots, A\zeta_a^{a-1}.$$

These parameters must have the property that, if they contain one primitive Mth root of unity for some M, then they contain all of them. Thus, by symmetry considerations, we find that the only possibilities are that $A = 1$ or $A = \zeta_{2a}$. However, we cannot have $A = 1$, as one of the denominator parameters for a hypergeometric group coming from an integral factorial ratio sequence will be 1, and the numerator and denominator parameters must be distinct. Thus, $A = \zeta_{2a}$. Similarly, for the denominator parameters we find that either B or C is 1, and so, without loss of generality, we assume that $C = 1$ and find that $B = \zeta_{2b}$ is the only possibility. Indeed, whenever $(a, b) = 1$ and a and b are not both odd, this does work and gives u_n of the form (5.1).

A similar argument works for the second case. □

We now examine the case where both a and b are odd.

Lemma 5.12. *Let $u_n = u_n(\mathbf{a}, \mathbf{b})$ and $\mathbf{u} = \mathbf{u}(\mathbf{a}, \mathbf{b}; z)$. Let $H(\mathbf{u})$ be the hypergeometric group associated to $\mathbf{u}$.*

(i) *If u_n is of the form (5.1) with a and b both odd, then $H(\mathbf{u})$ is a scalar shift by $-1 = e(1/2)$ of $H(\mathbf{u}')$, where*

$$u_n' = \binom{an}{bn}.$$

(ii) *If u_n is of the form* (5.2) *with a and b both odd, then $H(\mathbf{u})$ is obtained by taking a scalar shift of $H(\mathbf{u}')$ and reversing the numerator and denominator parameters, where*

$$u'_n = \binom{(a+b)n}{an}.$$

Proof. (i) Suppose that u_n is of the form (5.1). Then, as in Lemma 5.11, we have, for $P(x) = P(\mathbf{a}, \mathbf{b}; x)$ and $Q(x) = Q(\mathbf{a}, \mathbf{b}; x)$, that

$$\frac{P(x)}{Q(x)} = \frac{(x^a + 1)}{(x^b + 1)(x^{a-b} - 1)}.$$

This is not in the lowest terms, but it is in a convenient form for computing the scalar shift of $H(\mathbf{u})$. The scalar shift corresponds to multiplying each root of P and Q by -1, in which case we obtain polynomials P^* and Q^* with

$$\frac{P^*(x)}{Q^*(x)} = \frac{(x^a - 1)}{(x^b - 1)(x^{a-b} - 1)},$$

which very clearly come from $u'_n = \binom{an}{bn}$.

(ii) If u_n is of the form (5.2), then we proceed similarly, except that this time we find that

$$\frac{P^*(x)}{Q^*(x)} = \frac{(x^a - 1)(x^b - 1)}{(x^{a+b} - 1)},$$

which we can see comes from $u'_n = \binom{(a+b)n}{an}$, with the numerator and denominator parameters reversed. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Lemma 5.11 classifies all of those integral factorial ratios whose associated hypergeometric group is imprimitive, and so it remains to classify those with associated primitive hypergeometric groups. Beukers and Heckman have categorized all finite primitive hypergeometric groups, and so we can examine Table 8.3 in [**6**] to find all integral factorial ratios with associated primitive hypergeometric groups.

Specifically, if H is a primitive hypergeometric group that comes from an integral factorial ratio sequence, then H is either one of the entries in Table 8.3 in [**6**] or a scalar shift of one of the entries, possibly with the numerator and denominator parameters reversed. Moreover, if H has numerator parameters $\alpha_1, \ldots, \alpha_d$ and denominator parameters $\beta_1, \ldots, \beta_d$, then the polynomials $P(x) = \prod(x - e(\alpha_i))$ and $Q(x) = \prod(x - e(\beta_j))$ must have coefficients in $\mathbb{Z}$. Apart from line 1, there are 26 entries in the table with this property. Moreover, the entries in the table are listed so that, if the polynomials P and Q formed from the numerator and denominator

TABLE 1. *The three infinite families of integral factorial ratio sequences.*

Line #	$u_n(\mathbf{a}, \mathbf{b})$	d	${}_dF_{d-1}$ parameters	[**6**] line #
$1_{a,b}$	$[a+b]$ $[a,b]$	$a+b-1$	$[\frac{1}{a+b}, \frac{2}{a+b}, \ldots, \frac{a+b-1}{a+b}]$ $[\frac{1}{a}, \frac{2}{a}, \ldots, \frac{a-1}{a}, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b}]$	1
$2_{a,b}$	$[2a,b]$ $[a,2b,a-b]$	a	$[\frac{1}{2a}, \frac{3}{2a}, \ldots, \frac{2a-1}{2a}]$ $[\frac{1}{2b}, \frac{3}{2b}, \ldots, \frac{2b-1}{2b}, \frac{1}{a-b}, \frac{2}{a-b}, \ldots, \frac{a-b-1}{a-b}]$	None
$3_{a,b}$	$[2a,2b]$ $[a,b,a+b]$	$a+b$	$[\frac{1}{2a}, \frac{3}{2a}, \ldots, \frac{2a-1}{2a}, \frac{1}{2b}, \frac{3}{2b}, \ldots, \frac{2b-1}{2b}]$ $[\frac{1}{a+b}, \frac{2}{a+b}, \ldots, \frac{a+b-1}{a+b}]$	None

TABLE 2. *The 52 sporadic integral factorial ratio sequences.*

Line #	$u_n(\mathbf{a},\mathbf{b})$	d	${}_dF_{d-1}$ parameters	[6] line #
1	[12,1] [6,4,3]	4	$[\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}]$ $[\frac{1}{3},\frac{1}{2},\frac{2}{3}]$	37
2	[12,3,2] [6,6,4,1]	4	$[\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}]$ $[\frac{1}{6},\frac{1}{2},\frac{5}{6}]$	37
3	[12,1] [8,3,2]	6	$[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$ $[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}]$	45
4	[12,3] [8,6,1]	6	$[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$ $[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}]$	45
5	[12,3] [6,5,4]	6	$[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$ $[\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5}]$	46
6	[12,5] [10,4,3]	6	$[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$ $[\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}]$	46
7	[18,1] [9,6,4]	6	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4}]$	47
8	[9,2] [6,4,1]	6	$[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$ $[\frac{1}{6},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{5}{6}]$	47
9	[9,4] [8,3,2]	6	$[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$ $[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}]$	48
10	[18,4,3] [9,8,6,2]	6	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}]$	48
11	[9,1] [5,3,2]	6	$[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$ $[\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5}]$	49
12	[18,5,3] [10,9,6,1]	6	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}]$	49
13	[18,4] [12,9,1]	7	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$	58
14	[12,2] [9,4,1]	7	$[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	58
15	[18,2] [9,6,5]	7	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{3}{5},\frac{2}{3},\frac{4}{5}]$	59
16	[10,6] [9,5,2]	7	$[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	59
17	[14,3] [9,7,1]	7	$[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	60
18	[18,3,2] [9,7,6,1]	7	$[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	60
19	[12,2] [7,4,3]	7	$[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	61
20	[14,6,4] [12,7,3,2]	7	$[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$ $[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$	61
21	[14,1] [7,5,3]	7	$[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$ $[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{3}{5},\frac{2}{3},\frac{4}{5}]$	62
22	[10,6,1] [7,5,3,2]	7	$[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	62
23	[15,1] [9,5,2]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	63
24	[30,9,5] [18,15,10,1]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$	63
25	[15,4] [12,5,2]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$	64
26	[30,5,4] [15,12,10,2]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$	64
27	[15,4] [8,6,5]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{8},\frac{1}{6},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{5}{6},\frac{7}{8}]$	65
28	[30,5,4] [15,10,8,6]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{8},\frac{1}{3},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{2}{3},\frac{7}{8}]$	65

Table 2. *Continued*

Line #	$u_n(\mathbf{a},\mathbf{b})$	d	${}_dF_{d-1}$ parameters	[6] line #
29	[15,2] [10,4,3]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{10},\frac{1}{4},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{3}{4},\frac{9}{10}]$	66
30	[30,3,2] [15,10,6,4]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{5},\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{3}{4},\frac{4}{5}]$	66
31	[30,1] [15,10,6]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{4}{5}]$	67
32	[15,2] [10,6,1]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}]$	67
33	[15,7] [14,5,3]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$	68
34	[30,5,3] [15,10,7,6]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	68
35	[30,5,3] [15,12,10,1]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{12},\frac{1}{4},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{3}{4},\frac{11}{12}]$	69
36	[15,6,1] [12,5,3,2]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{12},\frac{1}{4},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{3}{4},\frac{11}{12}]$	69
37	[15,1] [8,5,3]	8	$[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}]$ $[\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8}]$	70
38	[30,5,3,2] [15,10,8,6,1]	8	$[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}]$ $[\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8}]$	70
39	[20,3] [12,10,1]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}]$	71
40	[20,6,1] [12,10,3,2]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{2}{3},\frac{11}{12}]$	71
41	[20,1] [10,8,3]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{8},\frac{1}{3},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{2}{3},\frac{7}{8}]$	72
42	[20,3,2] [10,8,6,1]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{8},\frac{1}{6},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{5}{6},\frac{7}{8}]$	72
43	[20,1] [10,7,4]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	73
44	[20,7,2] [14,10,4,1]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$	73
45	[20,3] [10,9,4]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	74
46	[20,9,6] [18,10,4,3]	8	$[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}]$ $[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$	74
47	[24,1] [12,8,5]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{5},\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{3}{4},\frac{4}{5}]$	75
48	[24,5,2] [12,10,8,1]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{10},\frac{1}{4},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{3}{4},\frac{9}{10}]$	75
49	[24,4,1] [12,8,7,2]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}]$	76
50	[24,7,4] [14,12,8,1]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}]$	76
51	[24,4,3] [12,9,8,2]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}]$	77
52	[24,9,6,4] [18,12,8,3,2]	8	$[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}]$ $[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}]$	77

parameters of an entry have coefficients in a field K, then the polynomials formed from a scalar shift of that entry will never have coefficients in a proper subfield of K, and so we only need to consider these 26 entries, along with line 1, which we mention momentarily.

As the denominator parameters for a hypergeometric group coming from a factorial ratio must contain a 1, there are only a finite number of scalar shifts of each entry that we need to consider. It is easy to enter the data from Table 8.3 in [6] into a computer program and to check each of the scalar shifts of each of these 26 entries. It turns out that there are 52 hypergeometric group parameter sets coming from the integral factorial ratios, and these are all given by these 26 entries and scalar shifts by -1 of these entries, possibly with the numerator and denominator parameters reversed.

It is easily seen that line 1 of Table 8.3 in [**6**] corresponds to the infinite family of binomial coefficient sequences

$$\frac{[(a+b)n]!}{(an)!(bn)!} \quad \text{with } \gcd(a,b)=1$$

and, as we have seen in Lemma 5.12, scalar shifts of this family by -1 yield factorial ratios of the forms (5.1) and (5.2) already considered. We need only to rule out other scalar shifts for this family.

The scalar shifts of this family that we need to consider are shifts by roots of unity of the form $e(-n/a)$, $e(-n/b)$, and $e\left(-n/(a+b)\right)$. The cases of multiplication by $e(-n/a)$ and $e(-n/b)$ are symmetric, and so we consider multiplication by $\zeta = e(-n/a)$. Now ζ is a primitive Mth root of unity for some M dividing a, and so we may as well assume that $\zeta = e(1/M)$. However, then $\zeta \cdot e(1/b)$ is a primitive bMth root of unity, as $(M,b)=1$. If $M \neq 2$, then some of the primitive bMth roots of unity are mapped to themselves by multiplication by $e(1/M)$, and so not all primitive bMth roots of unity can be obtained in this way. Thus, the polynomials P and Q obtained from this shift cannot have integer coefficients unless $M=2$, which is the case of a scalar shift by -1.

The case for multiplication by a root of the form $e\left(-n/(a+b)\right)$ is completely similar, since a, b, and $a+b$ are relatively prime in pairs.

Lemma 4.4 assures us that these must be all integer factorial ratio sequences in which the parameters have a greatest common divisor 1, giving us the 'only if' part of the theorem. □

6. *A listing of all integral factorial ratios with $L-K=1$*

Tables 1 and 2 contain a listing of all solutions to Problem 1.2 with gcd 1, organized as follows. The second column lists the parameters for $u_n(\mathbf{a},\mathbf{b})$. The parameter d of the third column is the dimension of the monodromy group of $\mathbf{u}(\mathbf{a},\mathbf{b};z)$, and the fourth column lists the specific parameters for ${}_dF_{d-1}$. With the exception of lines 2 and 3 of Table 1, all the entries have primitive monodromy groups, and so they have a corresponding entry in Table 8.3 in [**6**].

It should be noted that lines 1, 2, and 3 of Table 1 only correspond to solutions with gcd 1 when $\gcd(a,b)=1$.

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